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1985 J. Phys. G: Nucl. Phys. 11 359

(http://iopscience.iop.org/0305-4616/11/3/014)

J. Phys. G: Nucl. Phys. **11** (1985) 359–364. Printed in Great Britain

A simple relation for the Fermi function

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Received 18 July 1984, in final form 11 September 1984

Abstract. A simple relation for the Fermi function, correct to one per cent, is deduced using the tabulated values of Rose.

The probability $N(p)$ for the emission of a beta particle with momentum between p and $p + \mathrm{d}p$ per unit time is given (Fermi 1934) by

$$N(p)\,\mathrm{d}p = Cp^2(W_0 - W)^2\,\mathrm{d}p \tag{1}$$

where W_0 is the total energy of the electron corresponding to the end-point energy of the beta spectrum, W is the total energy corresponding to momentum p, and C is a constant. The energies and momenta are expressed in units of mc^2 and mc respectively, where m is the rest mass of an electron and c is the speed of light. According to equation (1) the distribution is proportional to p^2 for small momenta, whereas for large momenta it is proportional to $(W_0 - W)^2$. As the theory uses plane waves for the outgoing electron and neutrino, the effect of the Coulomb field on the emitted beta particle is neglected. But the Coulomb field of the daughter nucleus decelerates the negatron and accelerates the positron, thus altering the shape of the beta spectrum. The factor which accounts for the effect of the Coulomb field on the electron distribution is the Fermi function or Coulomb correction factor, $F(Z, W)$.

Without considering screening of the nuclear charge by orbital electrons, the relativistic Coulomb correction factor is given (Fermi 1934) by

$$F(Z, W) = \frac{2(1+S)}{((2S)!)^2}\,(2p\rho)^{2S-2}\,\mathrm{e}^{\pi\eta}\,|(S-1+\mathrm{i}\eta)!|^2 \tag{2}$$

where $S = (1-\alpha^2Z^2)^{1/2}$, $\rho = R/(\hbar/mc)$, R is the nuclear radius, p is the electron (positron) momentum, Z is the nuclear charge on the daughter nucleus, α is a fine structure constant and $\eta = \pm Ze^2/\hbar V$ where V is the speed of the beta particle far away from the nucleus and + (−) corresponds to an electron (positron).

There are a number of tabulations of the Fermi function (Rose 1955, Dzhelepov and Zyrianova 1956, Bhalla and Rose 1960, 1961, Bhalla 1964, Bühring 1965, Bahcall 1966, Suslov 1967, Blin-Stoyle and Nair 1967, Behrens and Bühring 1968, Behrens and Janecke 1969). All these calculations have used a uniform charge distribution for the nucleus. Furthermore, Behrens and Bühring (1972) have indicated that the assumption of a uniform charge distribution is perfectly satisfactory so long as the RMS radius is chosen correctly.

0305-4616/85/030359 + 06 $02.25

Several attempts have also been made to give simpler approximations to the Fermi function to make the calculations easy. If the electrons are treated as non-relativistic the correction factor is found to have the form (Mott and Massey 1933)

$$F(Z, W) = 2\pi\eta/(1 - \exp(-2\pi\eta)). \tag{3}$$

With this multiplying factor the electron distribution is found to be proportional to p rather than to p^2 for energies in the range $2\pi\eta \gg 1$. For low atomic numbers it is easy to see that equation (3) reduces to the form

$$F(Z, W) \simeq 1 + \pi\alpha ZW/p. \tag{4}$$

Hall (1950) suggested the following expression, which is found to have an accuracy within one percent:

$$F(Z, W) \simeq \frac{4(1+S)}{((2S)!)^2}(2p\rho)^{2S-2}(S^2+\eta^2)^{S-1/2}\exp[2\phi\eta - 2S + S/6(S^2+\eta^2)] \tag{5}$$

where $\varphi = \tan^{-1}(S/\eta)$. For $Z = 0$ this expression is found to give a value of 1.0046 instead of 1 for the Fermi function, indicating the error in the approximation.

Nilsson (1956) has derived an expression with an error of one-half percent at most:

$$F(Z, W) \simeq a\frac{W}{p} + \frac{C}{1 + d/p^2} \tag{6}$$

where $a = 2\pi\alpha Z$, $C = b - a$, $b = a/(1 - e^{-a})$, $d = \frac{1}{2}(b-1)$ and α is the fine structure constant.

Another approximation for the Fermi function, which is valid for high values of Z, has been given by Bethe and Bacher (1936):

$$F(Z, W) = F_N(Z, W)[W^2(1 + 4\gamma^2) - 1]^S \tag{7}$$

where $F_N(Z, W)$ is the non-relativistic expression given by equation (3), $\gamma = \alpha Z$ and $S = (1 - \alpha^2 Z^2)^{1/2} - 1$. This approximation is accurate to about one percent for atomic numbers as large as $Z = 84$.

While evaluating the integrated Fermi function in the allowed beta decay, Wilkinson (1970) has expanded the Fermi function in power series of αZ. Although this approximation gives an integrated Fermi function that agrees well with the exact function for high end-point energies, the deviation between the two increases systematically as the end-point energy decreases. Furthermore, it should be noted that the expansion coefficients are not simple functions of W and p.

In addition to altering the shape of the beta spectrum the Fermi function plays an important role in the modification of the inner bremsstrahlung (IB) spectral distribution that accompanies beta decay. To a first-order approximation the Fermi function is replaced by $1 + \pi\alpha ZW/p$ by Lewis and Ford (1956) in their IB theory. For allowed transitions they found that this factor does not significantly alter the correction to the KUB formula (Knipp and Uhlenbeck 1936, Bloch 1936) even when $F(Z, W)$ and $1 + \pi\alpha ZW/p$ differ from each other considerably. In order to achieve better accuracy in the calculation of the IB spectral distribution (Nilsson 1956, Ford and Martin 1969) and to provide an easy means for the construction of Fermi–Kurie plots we need a simpler but very accurate relation for the Coulomb factor.

A modified Fermi function, given by $G = Fp/W$, has been tabulated by Rose (1955) for each atomic number over a wide range of momenta. This has been very important data for

the experimental physicist for the determination of the end-point energy from the measured beta spectral distribution (Evans 1955) and also for other calculations where the Fermi function is involved.

The calculation of the Fermi function using equations (2) or (5) is tedious, and equations (6) and (7) are not simple either. Hence the need for greater dependence on the tabulated values. The same thing is true with regard to the integrals that appear in the evaluation of the IB spectral distribution and the integrated Fermi function. In the absence of a simple relation for $F(Z, W)$ for any specific electron momentum, the required value of the function is either interpolated from the tabulated values or calculated from the approximate expressions. Therefore, an attempt is made to obtain a simple relation for the calculation of the Fermi function for momenta $p \geqslant 25$ keV/c.

A plot of the square of $F(Z, W)$, obtained from the tabulated values of the modified Fermi function, as a function of $1/(W-1)$ is found to be linear from 25 keV to the end-point energy for any isotope. Figure 1 shows some typical plots for β emitters such as ^{32}P, ^{69}Zn, ^{137}Cs and ^{204}Tl. Therefore one can write

$$F(Z, W) = [A + B/(W-1)]^{1/2}. \tag{8}$$

The constants A and B, determined by a linear regression procedure, are given in table 1. A and B, however, become functions of the atomic number of the daughter nucleus. They are found to satisfy relations of the type

$$A = 1 + a_0 \exp(b_0 Z) \tag{9a}$$

$$B = aZ \exp(bZ) \tag{9b}$$

where a_0, b_0, a and b are constants. It is found that the constants a and b have different values in the two regions $Z \leqslant 56$ and $Z > 56$, whereas the constants a_0 and b_0 have the same values for $Z \geqslant 16$ as can be seen in figure 2.

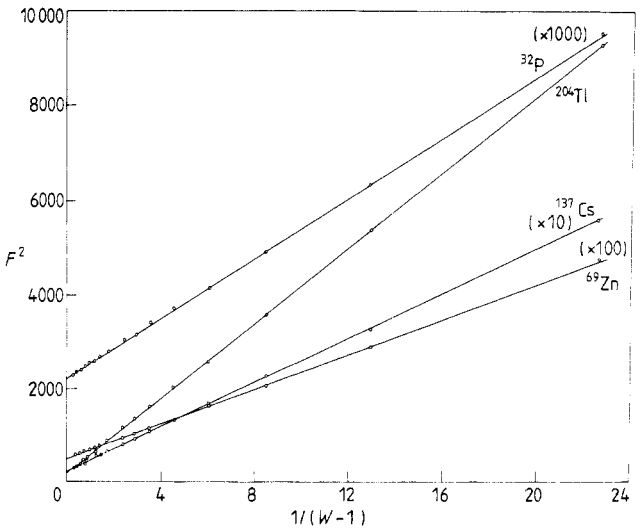

Figure 1. $F^2(Z, W)$ as a function of $1/(W-1)$.

Table 1. Values of A and B of equation (8).

Isotope	E_0 (keV)	Atomic no. of daughter	A	B
$^{6}_{2}$He	3500	3	1.1578	0.0208
$^{10}_{4}$Be	560	5	1.2985	0.0391
$^{14}_{6}$C	158	7	1.4864	0.0616
$^{19}_{8}$O	3200	9	1.5606	0.1015
$^{24}_{11}$Na	1390	12	1.8275	0.1711
$^{32}_{15}$P	1730	16	2.2243	0.3206
$^{35}_{16}$S	167	17	2.4496	0.3527
$^{36}_{17}$Cl	713	18	2.4981	0.4128
$^{45}_{20}$Ca	257	21	2.9595	0.5895
$^{49}_{21}$Sc	2001	22	3.0634	0.6814
$^{51}_{22}$Ti	2140	23	3.2342	0.7655
$^{69}_{30}$Zn	925	31	5.0709	1.8500
$^{75}_{32}$Ge	1190	33	5.7344	2.2849
$^{81}_{34}$Se	1510	35	6.4481	2.8135
$^{90}_{38}$Sr	546	39	8.2372	4.2458
$^{90}_{39}$Y	2274	40	8.6994	4.7116
$^{93}_{40}$Zr	600	41	9.3330	5.2025
$^{99}_{43}$Tc	292	44	11.1489	7.0753
$^{105}_{45}$Rh	565	46	12.9051	8.6289
$^{121}_{50}$Sn	393	51	17.8821	14.2645
$^{137}_{55}$Cs	1173	56	25.2007	23.6665
$^{170}_{69}$Tm	967	70	69.6298	103.7297
$^{185}_{74}$W	430	75	106.2196	179.7139
$^{204}_{81}$Tl	770	82	161.9399	403.5753
$^{210}_{83}$Bi	1170	84	180.5666	512.7254

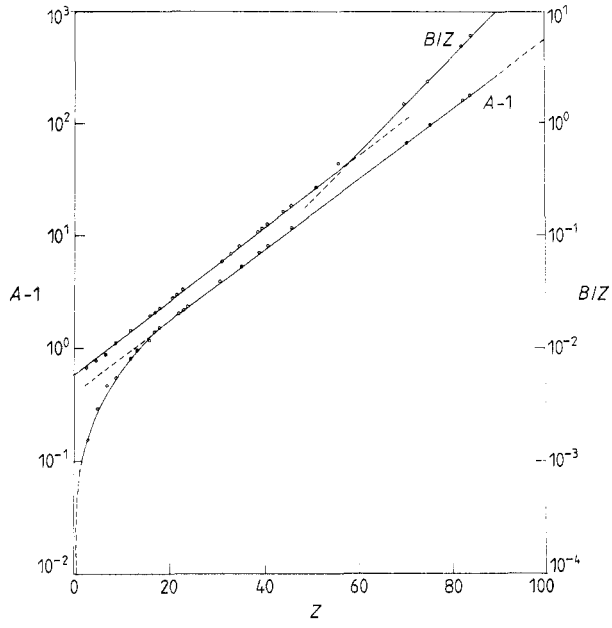

Figure 2. $\log(A-1)$ against Z and $\log(B/Z)$ against Z.

Using the values of A and B one can show from a linear regression that

$$a = 5.5465 \times 10^{-3} \qquad b = 76.929 \times 10^{-3} \qquad \text{for } Z \leqslant 56$$

and

$$a = 1.2277 \times 10^{-3} \qquad b = 101.22 \times 10^{-3} \qquad \text{for } Z > 56$$

whereas

$$a_0 = 404.56 \times 10^{-3} \qquad b_0 = 73.184 \times 10^{-3} \qquad \text{for } Z \geqslant 16.$$

For $Z < 16$ the parameter A can be represented by a linear relation $A = mZ + K$, where $m = 7.30 \times 10^{-2}$ and $K = 9.40 \times 10^{-1}$.

It can be seen that in the limit $Z = 0$ the values of A and B turn out to be 0.941 and 0, respectively. Hence from equation (8) we see that $F(0, W) = 0.970$ instead of unity, which shows an error of 3%.

For each isotope equation (8) is valid for momenta ranging from $p = 0.3$ to p_{max} corresponding to the end-point energy. When the values of $F(Z, W)$ are recalculated using equation (8), the tabulated values of Rose (1955) are reproduced within an error of one per cent.

We conclude, therefore, that equation (8) is a simple relation that reproduces Fermi function values of Rose with a reasonable accuracy. Hence it can be used conveniently in the construction of the Fermi–Kurie plot, in the evaluation of the integrated Fermi function in a closed form and in the calculation of the IB spectral distribution to obtain better results. The IB spectral distributions of Lewis and Ford (1956), Ford and Martin (1969) and others, modified by the present simple expression for $F(Z, W)$, will be communicated shortly.

Acknowledgments

The authors thank Dr D Krishnamurti, Professor and Head of the Department of Physics, University of Mysore for his support and encouragement.

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