LO, NLO, NNLO and resummed predictions

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Abstract. I provide a cursory review of the status of perturbation theory at various orders for observables measured at hadron colliders.

PERTURBATION THEORY

The many and varied tests of the Standard Model performed until now imply that any new physics is hiding well behind the decimal point [1]. Except for neutrino oscillations, no direct manifestation of new physics has appeared, so that, barring happy surprises, our keys to new discoveries are likely to be precise theory-experiment comparisons for suitable observables. High-luminosity hadron colliders such as the upgraded HERA and Tevatron and in particular the LHC, now under construction, can provide the precise measurements. Theorists must meet the challenge of providing the corresponding precise predictions. The tool for this is perturbation theory.

Perturbation theory allows the practitioner to parametrically increase the accuracy of theoretical predictions, by computing ever higher orders. At hadron colliders such efforts are done most profitably for QCD, because of the abundance of gluons and quarks and because the coupling is relatively large. For QCD observables, perturbation theory labels such N^{k}LO, $k = 0,1,2,\ldots$ and “resummed” mark the accuracy achieved by the practitioner.

Besides providing accurarcy, also the ability of perturbation theory to instruct should be kept in mind. If a finite order calculation works, in the sense that it behaves well as a perturbation series, and describes the data, we can claim we have understood something. Conversely, if the finite order calculations fail on both counts, then the way they fail (e.g. at large tranverse momentum of an observed particle) might point toward understanding (a need for resumming large tranverse momentum logarithms). Therefore, both verification and falsification are principles to hold dear in this endeavour. This is illustrated by the tale of the bottom quark transverse momentum spectrum as measured at the Tevatron. Cutting a long story [2, 3] short, it was shown [4, 5] that the problems that next-to-leading order theory had in describing the Tevatron data were actually symptoms of an improper use of perturbation in describing final state dynamics: an accurate resummation of FS radiation logs [6, 4], together with the consistent use in perturbation theory of the fragmentation function $D(z)$ [7] describing the transition from b-quark to B-meson removes most of the discrepancy.

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Best suited for comparisons between theory and experiment are fully differential cross sections

$$\frac{d^{3n}\sigma}{d^3p_1\ldots d^3p_n}, \quad 2\rightarrow \text{n scattering}\,.$$

The reason is that such cross sections allow numerical and thus flexible integration over regions of phase space where a particulcar experiment has acceptance. This leads to fairer comparisons, and almost always less theoretical uncertainty since more peripheral parts of phase space, where perturbation usually converges less well, are excluded from contributing. Moreover, different observables may be generated from such cross sections by selective integration over different variables ($p_T, y, \ldots$). E.g. from the fully differential cross section for the process $pp \rightarrow B + \text{jet} + X$ one may generate the observable

$$\int_{p^B_{T,min}}^{p^B_{T,max}} d^2p_T^{B-meson} \int_{y_{min}}^{y_{max}} dy^{jet} \int_{p^{jet}_{T,min}}^{p^{jet}_{T,max}} dp_T^{jet} \frac{d^4\sigma}{dy^{jet}\, d^2p_T^B\, dp_T^{jet}}\,.$$

The observable cross sections form the arena where experiment and quantum field theory are confronted. Their building blocks, computed in perturbative theory, are the scattering amplitudes M from which the cross section is calculated according to

$$\sigma = \frac{1}{2s}\overline{\sum_{s_i,c_i}}\int \frac{d^3p_1}{2E_1}\ldots\frac{d^3p_m}{2E_m}\delta^{(4)}(q_1+q_2-\sum_i p_i)\,|M(q_1,q_2,p_i,s_i,\mu,g(\mu))|^2 \qquad (1)$$

where q_1, q_2 are the initial state momenta and p_i, s_i, c_i are the final state momenta, spins and colors respectively. The coupling $g(\mu)$ runs as a function of the renormalization scale μ. The main trends in the last few decades in such calculations involved progressing from computing $|M|^2$ analytically to higher orders in g^2, for ever higher n, to computing M analytically, at fixed s_i so that the square and spin sum of M can be done by computer; from analytical to numerical phase space integrations; and most recently to interface such calculations with Monte Carlo generators at higher n and higner orders. These trends have led to impressive progress in accuracy for the theoretical description of hadron collider observables.

LO, NLO, NNLO

Assessments of how well these approximations compare with data, and what their convergence properties are must be made on a case by case basis. The sheer amount of important observables prevents me from doing a thorough survey of such assessments. Therefore I will only discuss these approximations briefly, mention some of their advantages and disadvantages, and indicate where the current challenges in each of them lie.

LO

To begin, we review some basic concepts involved in finite order calculations. In general, the perturbative expansion for an amplitude reads

$$M(g) = g^k M_0 + g^{k+1} M_1 + g^{k+2} M_2 + \dots$$

corresponding to the selection of diagrams in Fig. 1. Taking the square ($\alpha = g^2$) gives

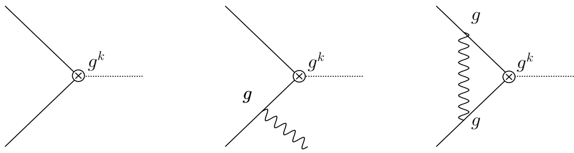

FIGURE 1. Perturbative expansion of amplitude

the series in Eq. (2).

$$|M(\dots g)|^2 = \underbrace{\underbrace{\alpha^k |M_0|^2}_{LO} + \alpha^{k+1} \left[|M_1|^2 + (M_2^\dagger M_0 + M_0^\dagger M_2 \right]}_{NLO} + \dots \tag{2}$$

What constitutes the LO and NLO approximations is schematically indicated. The LO description for an observable is defined by the condition that M_0 should produce the desired final state precisely, with no extra radiation, and with a minimal number of interactions.

The LO approximation has some pleasant properties. Being an absolute square, it is guaranteed to be positive. It is also finite, since all external momenta in $M_0(k_i, p_i)$ are fixed and different so that no propagator diverges. Consequently, LO cross sections have a straightforward interpretation as a scattering probability. Less pleasantly, their only μ dependence is in $\alpha(\mu)$ with no compensating term elsewhere in the expressions, so that, essentially, LO cross sections are unnormalized. As a consequence, the LO approximation is hard to falsify as far as the *size* of a signal is concerned. Falsification, indicating the need to go beyond the LO approximation, is best done for the *shape* of a signal, for which LO kinematics is often too constraining.

The present frontier in LO calculations lies with processes producing many particles and jets. Parton shower Monte Carlo programs based on $2 \to 2$ kinematics will of course produce such events, but will fail to describe their characteristics fully because they lack all the relevant matrix elements. LO descriptions of cross sections are relatively “simple”, they have the property of crossing symmetry, which helps reduce calculational effort. However, calculations of matrix elements for high particle/jet number production are still very hard. Momentum integrals, and even spin and color sums require clever strategies to perform them. Important recent progress has been achieved in matching the parton shower description to the matrix element description [8].

Another interesting recent innovation in this regard concerns the development of a new calculational method [9, 10], in which amplitudes are computed using maximal helicity violating subamplitudes [11] as building blocks.

The simplicity of the LO approximation allows a high level of automation in its use, via Monte Carlo programs. Among these are Feynman diagram based codes such as MadGraph/MadEvent [12, 13], Grace [14] and CompHEP [15]. There are also approaches that use efficient recursion relations rather than Feynman diagrams [16, 17], AlpGEN[18] and similar approaches [19]. For more information on the above approaces the interested reader might wish to consult the *Les Houches guidebook to Monte Carlo generators* [20].

NLO

We next describe some general characteristics of the NLO approximation for a partonic collisions. Consider quark annihilation into an off-shell photon of mass m (the Drell-Yan process). The three components of the NLO approximation, Born, virtual and real, are depicted in Fig. 2, where also some kinematic variables are defined. Notice that

$$s = (q_1 + q_2)^2, \quad p^2 = m^2, \quad w \equiv \frac{s-m^2}{s}$$

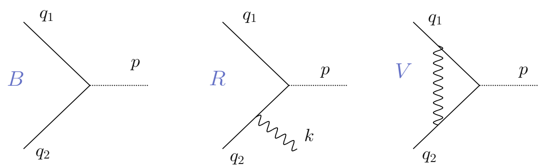

FIGURE 2. Born, virtual and real contributions to NLO cross section.

for the Born term and the virtual correction term $w = 0$. For the real emission term we have $w = 2(q_1 + q_2) \cdot k/s > 0$. A parton-level computation in $4 - \varepsilon$ dimensions yields, schematically

$$\begin{aligned}
\frac{d\sigma_B(m^2,w)}{dw} &= F_B(m^2)\,\delta(w) \\
\frac{d\sigma_V(m^2,w)}{dw} &= \alpha\left(\frac{-A}{\varepsilon^2}F_B(m^2) + F_V(m^2)\right)\delta(w)\,, \\
\frac{d\sigma_R(m^2,w)}{dw} &= \alpha\left(\frac{-1}{\varepsilon}\frac{K(w)}{w^{1+\varepsilon}}F_B(m^2) + F_R(m^2,w)\right), \quad K(0) = A\,.
\end{aligned}$$

The NLO approximation requires us to combine all three results. To this end, for this simple case, we use the identity

$$\frac{1}{w^{1+\varepsilon}} = -\frac{1}{\varepsilon}\delta(w) + \frac{1}{w_+} - \varepsilon\left(\frac{\ln w}{w}\right)_+ + \ldots$$

where on the right hand side we have used so-called plus distributions. They are well-behaved when integrated against smooth functions such as parton distribution functions.

Substituting this identity yields

$$\begin{aligned}\frac{d\sigma_B(m^2,w)}{dw} &= F_B(m^2)\,\delta(w)\\ \frac{d\sigma_V(m^2,w)}{dw} &= \alpha\left(\frac{-A}{\varepsilon^2}F_B(m^2)+F_V(m^2)\right)\delta(w)\,,\\ \frac{d\sigma_R(m^2,w)}{dw} &= \alpha\Big(\frac{A}{\varepsilon^2}F_B(m^2)\delta(w)-\frac{1}{\varepsilon}\frac{K(w)}{w_+}F_B(m^2)\\ &\qquad +K(w)\left(\frac{\ln w}{w}\right)_+F_B(m^2)+F_R(m^2,w)\Big)\,.\end{aligned}$$

We notice that the double poles in ε cancel, but that possibly annoying double logs have appeared in the last line (which is relevant for resummation, see below). The remaining $1/\varepsilon$ term can be absorbed via redefinition by the parton distributions functions (PDF's). As a result the $1/\varepsilon$ is effectively replaced by $\ln(\mu/m)$ in $d\sigma_R$. Then we have

$$\begin{aligned}\frac{d\sigma_{NLO}(m^2,w)}{dw} = F_B(m^2)\,\delta(w)+\alpha\Bigg(\ln\left(\frac{\mu}{m}\right)\frac{K(w)}{w_+}F_B(m^2)+\\ K(w)\left(\frac{lnw}{w}\right)_+F_B(m^2)+F_V(m^2)\delta(w)+F_R(m^2,w)\Bigg)\,.\end{aligned}$$

This was a simple case. Although conceptually the same, if the final state has many external lines the story is technically more complicated. How can the $1/\varepsilon$ terms cancel while keep all external lines fixed? To illustrate some common approaches, we consider the integral over w:

$$\begin{aligned}d\sigma = F_B(m^2)+\alpha\Bigg(\frac{-A}{\varepsilon^2}F_B(m^2)+F_V(m^2)+\\ \int_0^{w_{max}}dw\left[\frac{-1}{\varepsilon}\frac{K(w)}{w^{1+\varepsilon}}F_B(m^2)+F_R(m^2,w)\right]\Bigg)\end{aligned}$$

One well-known way the singularities can be cancelled is called *phase space slicing* [21, 22, 23, 24, 25, 26]. For small δ one may slice up the phase space volume for w near $w=0$ in order to isolate the poles from the radiative contribution and cancel them against those from the virtual contribution:

$$\begin{aligned}\int_0^{w_{max}}dw\frac{K(w)}{w^{1+\varepsilon}} &= \underbrace{\int_0^{\delta}dw\frac{K(0)}{w^{1+\varepsilon}}}_{analytical}+\underbrace{\int_{\delta}^{w_{max}}dw\frac{K(w)}{w}}_{numerical}\\ &= A\left(\frac{-1}{\varepsilon}+\ln\delta-\varepsilon\ln^2\delta\right)+\int_{\delta}^{w_{max}}dw\frac{K(0)}{w}\end{aligned}$$

The integral over w may now be carried out numerically, and thus flexibly. The constraint that δ is small may in fact be lifted [27, 28]. Mass factorization amounts again to replace the remaining $1/\varepsilon$ singularity with $\ln\mu/m$.

In the *subtraction method* [29, 30, 31], rather than slice up the w phase space, one attempts to find a clever $\tilde{K}(w)$ such that:

$$K(w) - \tilde{K}(w) \stackrel{w \to 0}{\to} c\,w\,,$$

but also

$$\int_0^{w_{max}} dw \frac{\tilde{K}(w)}{w^{1+\varepsilon}} = \frac{-A}{\varepsilon}\,,$$

so that

$$d\sigma = F_B(m^2) + \alpha\Big(F_V(m^2) + \int_0^{w_{max}} dw \left[\left(\frac{-1}{\varepsilon}\right) \frac{K(w) - \tilde{K}(w)}{w} F_B(m^2) + F_R(m^2, w)\right]\Big)\,. \quad (3)$$

Mass factorization operates as before. The parton level results, once they are made finite, must be convoluted with PDF's

$$d\sigma_H = \Phi(\mu) \otimes d\sigma(\mu/m, \alpha(\mu))$$

here expressed via the parton flux Φ. We remark that while in the partonic NLO cross section the scale dependence is present through NLO, in the parton flux $\Phi(\mu)$ as well as in $\alpha(\mu)$, the μ dependence is usually present to all orders, so that scale dependence remains, at orders beyond the approximation used.

To assess the value of doing a NLO calculation for observable O, let us examine its general structure.

$$O_{NLO} = \alpha^k(\mu)A + \alpha^{k+1}(\mu)\left[B + kb_2 \ln\frac{\mu}{m} A\right] + \ldots$$

This structure suggests the first benefit: that the observable's normalization is better understood, since the scale uncertainty is reduced by explicitly scale dependent terms at order α^{k+1}. The second benefit is that because extra radiation is allowed beyond the minimal final state, a better modeling of the latter is achieved. Less welcome might be that cross sections, in certain regions of phase space, can become negative, since the matrix element is no longer an absolute square, see Eq. (2). Since this implies that the corrections are larger than the Born term, this can again point to the need to go beyond the NLO approximation.

The present NLO frontier lies in varied terrain. The most difficult to traverse involves increasing the number of external particles (≥ 6). Here loop diagrams are the main bottleneck [32, 33, 34]. Right behind the frontier, valuable settlement efforts are creating a consistent collection of NLO codes for many important signals and their backgrounds (e.g. MCFM project [35]). Other patches of difficult terrain are associated with processes that require the practictioner to take into account extra factorization conditions. For example, relatively few [36, 37] NLO calculations have been done for quarkonium production cross sections, done in the factorized context of NRQCD [38]. Another example is formed by various variable flavor number schemes calculations [39, 40, 41, 42, 43] done in the context of factorizations with heavy quarks in the initial state [44], relevant for processes producing heavy quarks at high energy.

Other important progress has been made [45, 46, 47] in the consistent matching of NLO calculations to parton shower Monte Carlo's. The combinations combine the benefits (and of course some of the drawbacks) of both approaches, and will be of great value to Tevatron and LHC analyses.

NNLO

The calculation of hadron-collider scattering observables in NNLO approximation is fiendishly difficult. To explain why this goal is nevertheless being pursued vigorously, I quote an example by Glover. Consider [48] the transverse single-jet energy distribution in $p\,\bar{p}$ collisions. Its structure at NNLO is

$$\begin{aligned}\frac{d\sigma}{dE_T} &= \alpha_s^2(\mu)A_2 + \alpha_s^3(\mu)\left[A_3 + 2b_2A_2\ln\left(\frac{\mu}{E_T}\right)\right] \\ &+ \alpha_s^4(\mu)\left[A_4 + 3b_2\ln\left(\frac{\mu}{E_T}\right)A_3 + (3b_2^2\ln^2\left(\frac{\mu}{E_T}\right) + 2b_3\ln\left(\frac{\mu}{E_T}\right)A_2\right],\end{aligned}$$

where the coefficients A_2, A_3 are known, A_4 is still unknown. Even without knowing

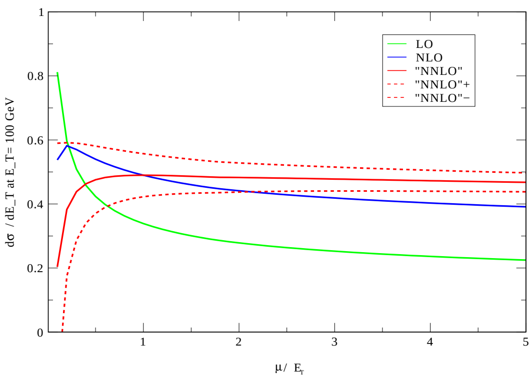

FIGURE 3. Scale ($\mu_R = \mu$) dependence of the LO (green), NLO (blue) and 3 NNLO (red) approximations to the single jet inclusive distribution at the Tevatron.

A_4 it is clear from Fig. 3 that the scale uncertainty of the cross section is significantly reduced, allowing a much more accurate and thereby fruitful confrontation with data. A_4's value is of course important to know the actual size of the distributions.

So far, NNLO calculations for hadron colliders are done for some important inclusive $2 \to 1$ processes, such as Drell-Yan ($q\,\bar{q} \to \gamma^*, W, Z$) [49, 50], polarized Drell-Yan (relevant for the RHIC collider) [51], the deep-inelastic structure functions [52, 53, 54, 55], and Higgs production in gg fusion [56, 57, 58]. Recently also related differential cross sections have been determined to NNLO [59, 60]. A missing ingredient to all these results was a complete set of NNLO PDF's, the construction of which requires the NNLO evolution kernels, which in turn require knowing the corresponding anomalous dimensions $\gamma(N)$ (related to the QCD splitting functions $P(z)$ via $\gamma(N) = -\int_0^1 dz\, P(z)$) to three loops:

$$\frac{d \ln \phi(N,\mu)}{d \ln \mu} = \alpha_s(\mu)\gamma_1(N) + \alpha_s^2(\mu)\gamma_2(N) + \alpha_s^3(\mu)\gamma_3(N) \tag{4}$$

Moreover, the NNLO anomalous dimensions are required for factorization scheme invariance of the calculation.

In a landmark achievement, the computation of the three-loop splitting functions has been recently completed [61, 62], providing the key to fully NNLO observables. The calculation involved about 10,000 Feynman diagrams and very intensive use of, and extensions to the computer algebra program FORM [63]. With this much complexity, it was of vital importance that the method used allowed checks on a per diagram basis with earlier calculations [64, 65, 66] for a number of fixed moments, done with an entirely different method. The results shown in Fig. 4 indicate that we now know the size of the

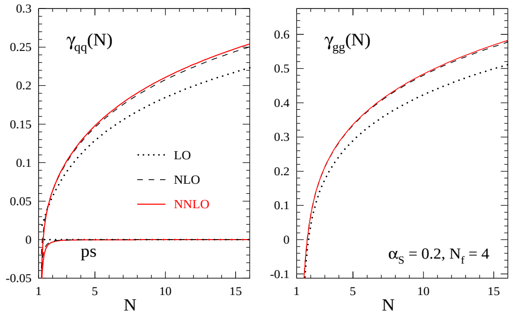

FIGURE 4. Convergence of NNLO approximation for splitting functions.

anomalous dimensions to sufficient precision.

Another universal ingredient of the NNLO approximation, the (three-loop) QCD β-function, has been known for some time [67, 68]. With the anomalous dimensions, β-function and key processes now known to NNLO, constructions of NNLO PDF's are possible using global analysis involving NNLO DY and DIS calculations. There is one other necessary ingredient: the perturbative matching conditions on the flavor thresholds. Fortunately, these are also known to NNLO [69]. In the $\overline{\mathrm{MS}}$ scheme these conditions imply that parton densities are discontinuous across thresholds, an effect that also seen for the running coupling at NNLO in this scheme [70, 64, 71]. The frontier in NNLO calculations lies now with $2 \to 2$ processes. A global, heroic effort is ongoing to determine the required ingredients (two-loop $2 \to 2$, one-loop $2 \to 3$, tree-level $2 \to 4$) and understand their infrared structure, in order to construct a workable substraction method. Although the many important contributions to this effort are too numerous to cite in this cursory review, the breakthrough that jump-started this effort, the calculation of two-loop box scalar integrals [72, 73], should be mentioned:

NNNLO

There are even a few NNNLO quantities for hadron colliders observables known exactly, the NNNLO QCD β-function [74] already for some time. Recently, the NNLO coefficient functions for F_L have been computed [75], leading us to anticipate the NNNLO coefficient functions for F_2 in the near future.

RESUMMATION

"Resummation" is shorthand for all-order summation of classes of potentially large terms in perturbation theory. To get an impression of what resummation is and does, consider $d\sigma$, a quantity with the schematic perturbative expansion

$$d\sigma = 1 + \alpha_s(L^2 + L + 1) + \alpha_s^2(L^4 + L^3 + L^2 + L + 1) + \dots \tag{5}$$

where α_s is the coupling, also serving as expansion parameter and L is some logarithm that is potentially large. In our discussion we focus on gauge theories, and on the case with at most two extra powers of L per order, as Eq. (5) illustrates. An extra order corresponds to an extra emission of a gauge boson, the two ("Sudakov") logs resulting from the situation where the emission is simultaneously soft and collinear to the parent particle direction.

Denoting $L = \ln A$, one might ask what A is. In fact, A depends on the quantity $d\sigma$. For example, for a thrust (T) distribution $A = 1 - T$, while for $d\sigma(p\bar{p} \to Z + X)/dp_T^Z$ at small p_T^Z, $A = M_Z/p_T^Z$. In this case, the observed Z boson transverse momentum results from *recoil* against unobserved parton emissions, It should be pointed out already here that A is not necessarily constructed out of measured variables but can also be a function of unobservable partonic momenta to be integrated over. E.g. for inclusive heavy quark production A could be $1 - 4m^2/x_1x_2S$ in hadron collisions with energy $\sqrt{S}$, where x_1, x_2 are partonic momentum fractions. Near *threshold* is L numerically large, endangering the behaviour of the series (5).

In these cases, when L is large, the terms containing L can often be "resummed" to all orders, i.e., organized into a form whose expansions reproduces the terms in Eq. (5). The gathering of the problematic terms into an analytic form might provide a remedy to the bad perturbative behavior, and thereby extend perturbation theory's predictive power to situations in which its validity at first might seem problematic.

Such resummations are usually further specified by the type of logarithm that is resummed. Referring to the above, one talks of recoil resummation, or threshold resummation, or even joint (recoil and threshold) resummation.

The resummed form of $d\sigma$ may be written schematically as

$$d\sigma_{res} = C(\alpha_s)\,\exp\left[Lg_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \dots\right] + R(\alpha_s) \tag{6}$$

where $g_{1,2,\dots}$ are computable functions. The series $C(\alpha_s)$ multiplies the exponential, and $R(\alpha_s)$ denotes the remainder.

We can make some remarks about Eq. (6). First, the residual series $C(\alpha_s), R(\alpha_s)$ are without logs, and therefore (presumably) better-behaved. The dependence on the logarithm has moved into the exponent, which is now a series in α_s, and under better control. This is the main merit of resummation. The formula also has a certain predictivity, in the sense that the one-loop leading logarithmic term, after exponentiation, predicts the leading logarithmic terms at all orders, and so forth for subleading terms. Second, the exponential in the resummed form reflects, in a sense, the Poisson statistics of independent gauge boson emissions. Third, to implement energy or momentum conservation in a convenient way, the L in (6) is often not the logarithm of a (observed or unobserved) momentum space variable (say, p_T), but rather of a conjugate variable (impact parameter b) resulting from a Fourier or other integral transform. An expression like (6) may then be evaluated numerically and used phenomenologically, after an appropriate inverse transform, but it should be mentioned that this is not always an unambiguous procedure, in particular for QCD: the all-order resummation can introduce severely singular infrared behaviour into $d\sigma_{res}$ that is not present in finite order approximations. Therefore, a resummed result must, in such cases, be specified together with a prescription how to handle this singular behaviour numerically. Although a nuisance in this sense, such ambiguities can in fact be interesting in themselves because they take the form of power corrections. The study of such ambiguities thereby provides access to these important and insufficiently studied corrections.

To illustrate this point, consider threshold resummation. It begins with defining a threshold (elastic limit), $w = 0$. The threshold depends on the observable, e.g. for heavy quark production, for the inclusive cross section one can use $w = s - 4m^2 = 0$, for p_T distribution one can use $w = s - 4(m^2 + p_T^2) = 0$, and for the double-differential distribution $w = s + t + u - 2m^2 = 0$. The large logarithm to resum is $\ln w$. To resum $\ln(w)$ effects, it is best to first pass to a conjugate space via a Laplace (or Mellin) transform $\int_0 dw \exp(-wN)$. Notice that the limit $w \to 0$ corresponds to $N \to \infty$. The large log is then, in conjugate space, $\ln N$. After resummation, one must undo the transform via

$$\int_C dN \exp(wN) f(N) \tag{7}$$

with C an appropiately chosen contour, such as indicated in Fig. 5. A choice for handling

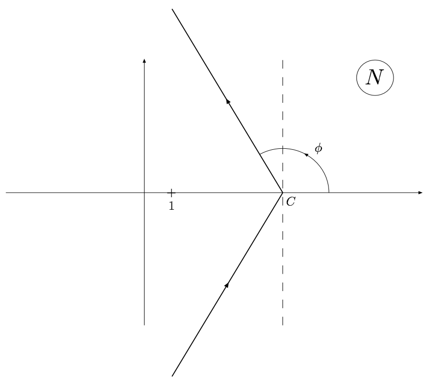

FIGURE 5. Contour for inverse Laplace or Mellin transform

the singular behavior mentioned above can be translated into the choice for the contour in Fig. 5.

As we have seen, specifying the theoretical accuracy for a perturbative series such as Eq. (5) involves stating whether only (leading order (LO)) the lowest order term has been kept, or in addition (next-to-leading order (NLO)) the $\mathscr{O}(\alpha_s)$ term, etc. The analogue for the resummed form (6) involves stating whether only g_1 is kept (leading logarithmic (LL) approximation), or in addition g_2 (next-to-leading logarithmic (NLL)) is kept, etc. Note that an increase in the logarithmic accuracy must go along with the inclusion, without double counting, of more terms in the $C(\alpha_s)$ series. This procedure is called matching. Just as one may parametrically and systematically increase the accuracy of the perturbative approximation (5) by including ever higher order terms, one may do so for the resummed expression by including ever more terms in the exponent, together with appropriate matching.

The benefits of resummation are *(i)* enhancement of predictive power, *(ii)* an increase in theoretical accuracy (e.g. reduction of scale uncertainty), and *(iii)* guidance from QCD resummation ambiguites toward non-perturbative effects. Present trends strive to *(i)* increase accuracy by including higher order terms in exponent, and match accordingly *(ii)* extend resummation to more-differential cross sections, and new classes of terms (other logs, constants..), and *(iii)* further develop resummation as a precise and practical tool in phenomenology.

Threshold, recoil and joint resummation

To appreciate the results discussed in what follows, I provide, for illustrative purposes and without derivation, the general forms for the threshold, recoil and joint-resummed cross sections for the Drell-Yan process, in which a vector boson of mass Q, in the latter two cases also at fixed transverse momentum $\mathbf{Q}_T$, is produced.

The threshold-resummed Drell-Yan cross section, mass factorized in the $\overline{\mathrm{MS}}$ scheme, takes the following general form [76, 77]

$$\begin{aligned} \frac{d\sigma_{a\bar{a}}}{dQ^2}(Q,z) &= H(Q)\int \frac{dN}{2\pi i} z^{-N} \times \\ & \exp\left[\int_0^1 \frac{dw}{w}\left(e^{-Nw}-1\right)\int_1^w \frac{d\lambda}{\lambda} A_{a\bar{a}}(\alpha_s(\lambda Q)) + D_{a\bar{a}}(\alpha_s(wQ))\right] + Y_{a\bar{a},th}\,, \end{aligned} \tag{8}$$

expressed as an inverse transform with an appropiate contour for the N integral. The functions A and D depend on α_s only, with A controlling leading and some subleading logarithms, and D the remaining subleading logarithms. The function Y is analogous to R in Eq. (6).

The recoil-resummed partonic cross section for electroweak boson production in hadronic collisions, takes the form of an inverse transform over the impact parameter

b [78, 79, 80]

$$\frac{d\sigma_{ab}}{dQ^2\,d^2\mathbf{Q_T}}(Q,\mathbf{Q_T}) \;=\; H_{cd}(Q)\int \frac{d^2b}{(2\pi)^2} e^{i\mathbf{Q_T}\cdot\mathbf{b}}(Q)C_{c/a}(b)\otimes C_{d/b}(b)\otimes$$
$$\exp\left[-\int_{1/b}^{Q}(dq/q)A(\alpha_s(q))\ln(Q/q)+B(\alpha_s(q))\right]+Y_{ab,rec}\,. \quad (9)$$

The convolution is in collinear momentum fractions. The large logarithm that is resummed here is $\ln b$. Notice the Sudakov *suppression* in the exponential at large b, which will appear in various figures below as a suppression of the cross section at very small Q_T. Note also that carrying out the inverse b transform requires a choice how to handle the large b part of the integral, as mentioned in the introduction. A number of popular choices for such non-perturbative input have emerged over the years. For the precise form of the resummed cross section used for each recoil-resummation study discussed in the following sections, as well as the choice of non-perturbative input, one should consult the relevant reference. E.g. a number of studies use a form similar to (9) expressed in momentum space [81], not requiring an inverse b transform.

Finally, the joint-resummed expression for electroweak production may be written, equally schematically, as [82, 83]

$$\frac{d\sigma_{a\bar{a}}}{dQ^2\,d^2\mathbf{Q_T}}(Q,\mathbf{Q_T},z) \;=\; H(Q)\int\frac{dN}{2\pi i}z^{-N}\int\frac{d^2b}{(2\pi)^2}e^{i\mathbf{Q_T}\cdot\mathbf{b}}$$
$$\times\tilde{C}_a(N,b,Q,\mu)e^{E_{a\bar{a}}(N,b,Q,\mu)}\tilde{C}_{\bar{a}}(N,b,Q,\mu)+Y_{a\bar{a},joint}\,. \qquad (10)$$

Notice the combined N and b dependence in the various functions. This formalism resums $\ln N$ and $\ln b$ logarithms simultaneously. In the limit of small N or large b the function $\tilde{C}$ reduces to C, at least up to a certain accuracy.

Let us make a few general obserbation about threshold resummation that illustrate where it often leads to cross section enhancements, where recoil resummation led to suppression. In N space Sudakov suppression takes the form

$$\lim_{N\to\infty}\sigma_H(N)\sim\exp(-\ln^2 N)\,. \qquad (11)$$

This is indeed true for the hadronic cross section σ_H, but recall that we are resumming the corrections to the *partonic* cross section, which is the ratio of the hadronic section and the squared parton distribution function

$$\sigma(N)=\frac{\sigma_H(N)}{\phi^2(N)}\,. \qquad (12)$$

The parton distribution themselves are also Sudakov suppressed: $\phi(N)\sim\exp(-\ln^2 N)$, so that

$$\lim_{N\to\infty}\hat{\sigma}(N)\sim\exp(+\ln^2 N)\,. \qquad (13)$$

In words: parton distributions are too stingy with gluon radiation, from including too many virtual gluons, so that in the ratio (12) the net effect is enhancing. Note however that jets produced at the hard scattering are also Sudakov suppressed near threshold. A sufficiently strong suppression from such final state jets can overcome the enhancement from initial state jets.

In joint resummation threshold enhancement and recoil suppression compete for dominance and the net effect can depend on the kinematics.

Having formed an impression of the concepts and some of the technicalities behind resummation, let us next turn to various studies involving the resummations mentioned. Since this review merely touches upon the key points of these studies, is certainly not exhaustive in discussing all their merits, nor in listing all relevant literature, I refer the interested reader to the original papers and references therein for more details.

Threshold resummation in Drell-Yan

The Drell-Yan process has a distinguished history as the theoretical laboratory for testing QCD resummation ideas [76, 77]. The present level of accuracy in threshold resummation is NNLL [84], achieved by matching to the NNLO calculations of Refs. [49, 50].

The most accurate studies of threshold resummation for Drell-Yan are performed in Ref. [84]. Representing the partonic resummed cross section in moment space one finds

$$\sigma_{DY}(N,Q^2) = H(Q)\exp\left[G_{DY}(N,Q)\right] \tag{14}$$
$$G_{DY} = 2\ln N g_1(2\lambda)+g_2(2\lambda)+\alpha_s g_3(2\lambda)+\ldots \tag{15}$$
$$\lambda = \beta_0\alpha_s\ln N\,. \tag{16}$$

Here

$$g_1(\lambda)=\frac{C_F}{\beta_0\lambda}\left[\lambda+(1-\lambda)\ln(1-\lambda)\right]. \tag{17}$$

Results are shown in Fig. 6 for the convergence properties when increasing the logarithmic accurcay of the exponent, and of the hadronic K factor.

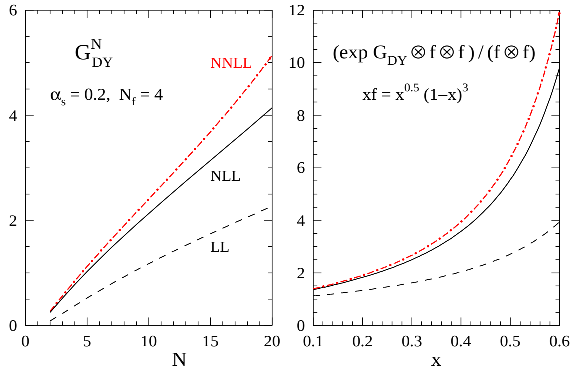

FIGURE 6. Convergence behavior of Drell-Yan partonic and hadronic cross section.

One observes good convergence as the logarithmic accuracy of the resummation is increased. For the inverse Mellin transform, required to compute the hadronic cross section in momentum space, the minimal prescription [85] was used: the contour in Fig. 5 is chosen such that the intercept C is to the left of the singularity at $\lambda = 1/2$ in Eq. (17).

Threshold resummation in Higgs production

Although the Higgs boson is not (yet) discovered, the QCD corrections to its inclusive production cross section have received a large amount of attention in recent years, making it now perhaps the most accurately computed cross section in QCD.

The NLO calculations, after having been done first in the heavy m_t limit [86, 87] were carried out with full m_t dependence in Ref. [88]. The large size of the corrections prompted a study [89] involving partial NLL threshold resummation for the Higgs cross section, which also showed that heavy m_t limit is in fact valid for $m_H \gg m_t$. Threshold resummation is relevant even for light Higgs at the LHC, because the gluon flux distribution will produce Higgs mostly near partonic threshold. This study also noticed the importance of $\ln^i N/N$ terms, of purely collinear origin.

Its exact corrections have now been computed, in the large m_t limit, to NNLO by various groups [57, 56, 58]. The threshold-resummed cross section is similar in form to Eq. (14), and all functions $g_{1,2,3}$ are known [90]. In Ref. [90] the full NNLL threshold-resummed Higgs cross section, matched carefully to NNLO, and including the $\ln^i N/N$ terms was constructed and uncertainties estimated. Results are shown in Fig. 7 for the K factor, defined with respect to the LO cross section at a fixed scale. The resummed cross

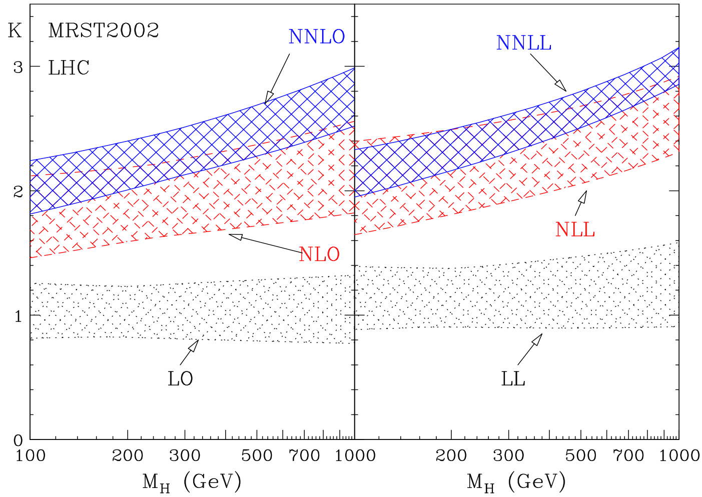

FIGURE 7. K-factor for Higgs production at the LHC at various orders

section has smaller scale uncertainty than the fixed order one, as indeed should happen generically [91]. The NNLO corrections and the threshold resummation together show that the perturbative series for the Higgs production cross section is considerably better-behaved that at first feared.

Recoil resummation in Higgs production

Recoil resummation attempts to predict the *shape* of the Higgs boson transverse momentum spectrum. A number of studies [46, 47, 92, 93, 82, 94, 95] have now been performed for this observable. In the 2003 Les Houches workshop [96] a comparison was made between the various calculations, including in the comparison the spectra as produced by PYTHIA [97] and HERWIG [98]. The results of the various studies are shown in Fig. 8. We see that the analytic resummations, the most accurate (NNLL) one

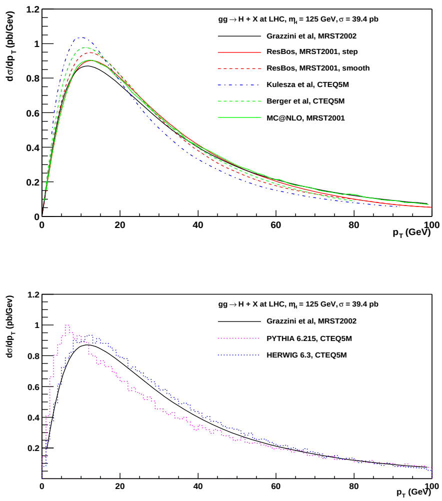

FIGURE 8. Predictions for the production of a 125 GeV Higgs boson at the LHC, all normalized to the same cross section.

of which is that of [93], followed by that of [99], are quite consistent with eachother. Also the shape from HERWIG's prediction is consistent with the analytic shapes, while that of PYTHIA is considerably softer. For a more complete discussion see the relevant contributions in Ref. [96].

All curves for the analytic resummations are based on versions of the recoil-resummed formula (9), except the one labelled Kulesza et al, which is based on the joint resummation formula Eq. (10). The joint-resummed shape is slightly softer that the recoil shapes.

Joint resummation for W and Z production

These same authors [83] also derived the recoil spectrum using joint resummation for the W and Z bosons at the Tevatron, performing not only the N and but also the b integral with minimal prescription [100], vitiating the need for any non-perturbative input. As the dash-dotted line in Fig. 9 shows, the result is in strikingly good agreement with CDF (and D0) data. The agreement is even slightly improved by nevertheless including a

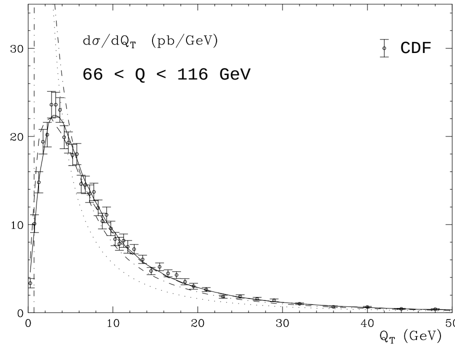

FIGURE 9. Recoil spectrum from joint resummation for W,Z production at the Tevatron

small non-perturbative component (solid line).

Threshold resummation in heavy quark production

An observable based on $2 \to 2$ kinematics for which threshold resummation has been studied extensively is the inclusive top quark cross section at the Tevatron. For threshold resummation of this observable it is sufficient to define threshold as $s = 4m^2$, with m the top mass, although other choices are possible as well. The NLL form of the resummed cross section can again be written as an inverse Laplace transform

$$\sigma_{ij}(w) = \int_C dN \exp(wN) \exp\left[\ln N g_1(\alpha_s \ln N) + g_2(\alpha_s \ln N)\right]$$

where ij stands for either the $q\bar{q}$ production subprocess (dominant at the Tevatron), or the gg production subprocess. While earlier studies [101, 102, 85] gave progressive understanding about how to threshold-resum this observable, the lack of proper understanding of the soft radiation, included in g_2, prevented reaching NLL accuracy. In Refs. [103, 104] it was shown that the resummation of the soft function and the effects of coherent soft gluon emission from multiple underlying color antennas it contained could be controlled by renormalization group and anomalous dimensions for special composite operators constructed out of Wilson lines. For the inclusive NLL threshold-resummed top quark cross section this was implemented in Ref. [105] and applied in a recent study providing the best estimates for size and uncertainties of the top quark production cross section [106].

At this point we can mention another use of resummed formulae: providing a controlled estimate of uncalculated higher orders by expansion. We consider the threshold-resummed double-differential cross section for top quark pair production, which requires the threshold $s+t+u=0$. Using the formalism of Ref. [107], the result has the form [108]

$$\begin{aligned} \frac{d^2\sigma_{ij}(w,A,B)}{dA\,dB} &= \int_C dN \exp(wN) \times \\ & \exp\left[\ln N g_1(\alpha_s \ln N, A, B) + g_2(\alpha_s \ln N, A, B)\right] \end{aligned} \tag{18}$$

with A,B define either in single-particle inclusive kinematics $(A,B)=(t,u)$, or in pair invariant mass kinematics $(A,B)=(M_{t\bar{t}},\cos\theta)$ with θ the c.m. scattering angle. The estimate now proceeds in the following straightforward way: (*i*) expand to NLO and NNLO with appropiate matching, (*ii*) use NLO to judge the quality of the NNLO estimate. The result is a set of formulae for all NNLO corrections of order $\alpha_s^2 \ln^i w$ with $i=4,3,2$. These formulae have been used to estimate NNLO corrections to the inclusive cross section [108] (after integration over A,B) for both top quark production at the Tevatron and b quark production for the HERA-B experiment, as well as differential cross sections [109]. Even though these are only estimates, they contain enough higher order information for a significant reduction of scale dependence.

Some recent developments

We finally turn some to a few recent developments in the methodology of resummation. Here we discuss the extension of the class of terms that can be resummed from logarithms to constants, as represent by the terms labelled "1" in Eq. (4). The resummations of such terms, once put on a theoretically sound basis, could have significant phenomenological consequences Ref. [89].

The first evidence that the dominant non-logarithmic perturbative contributions could be exponentiated goes back to [110], where it was shown that the partonic Drell-Yan cross section in the DIS factorization scheme contains the ratio of the timelike to the spacelike Sudakov form factor: large perturbative contributions are left over in the exponentiated form of this ratio after the cancellation of IR divergences. This observation was made more precise in Ref. [76]. There, the resummation of threshold logarithms for the Drell-Yan process was proven to all logarithmic orders, making use of refactorization: the Mellin transform of the cross section is expressed near threshold, approached by letting the Mellin variable N grow very large, as a product of functions, each organizing a class of infrared and collinear enhancements; the refactorization is valid up to corrections which are *suppressed* by powers of N at large N, so that terms independent of N (constants) can also be treated by the methods used to resum logarithms of N. In Ref. [111] the results of Refs. [76, 112] were exploited to show that for processes which are electroweak at tree level the exponentation of N-independent terms is in fact complete. In the $\overline{\mathrm{MS}}$ scheme e.g. one finds

$$\begin{aligned}\sigma_{\overline{\mathrm{MS}}}(N) &= \left|\frac{\Gamma(Q^2,\varepsilon)}{\Gamma(-Q^2,\varepsilon)}\right|^2 \left(\frac{\Gamma(-Q^2,\varepsilon)}{\phi_v(Q^2,\varepsilon)}\right)^2 \exp\left[\mathscr{F}_{\overline{\mathrm{MS}}}(\alpha_s)\right] \exp\left[\int_0^1 dz \frac{z^{N-1}-1}{1-z}\right. \\ &\times \left.\left\{2\int_{Q^2}^{(1-z)^2Q^2} \frac{d\xi^2}{\xi^2} A\left(\alpha_s(\xi^2)\right) + D\left(\alpha_s\left((1-z)^2Q^2\right)\right)\right\}\right] . \end{aligned} \tag{19}$$

All the functions appearing in Eq. (19) are in fact pure exponentials and can be explicitly evaluated at two loops by matching with the complete two-loop calculation of Ref. [50]. The refactorization analysis at the level of constants offers however an alternative, and often simpler method to determine the two-loop coefficients in these expressions by using the fact that real and virtual contributions in the factorized expressions can be made separately finite.

It should be pointed out that the exponentiation of N–independent terms does not have the predictive power of the standard resummation of threshold logarithms. In that case, typically, an entire tower of logarithms can be exactly predicted to all orders by performing just a low order calculation. Here, on the other hand, functions such as $\mathscr{F}_{\overline{\mathrm{MS}}}$ receive new nontrivial contribution at each perturbative order. The exponentiation pattern is nonetheless nontrivial, new equations emerge, and higher order terms predicted by the exponentiation can be considered representative of the size of the complete higher order correction [111].

A novel approach to resummation is one that removes the theorist from the game [113], an approach at present mostly directed at the resummation of event shape variables. Event shapes and jet resolution parameters (final-state 'observables') measure the extent to which the energy-flow of the final state departs from that of a Born event. Their study has been fundamental for measurements of the strong coupling [114, 115] as well as the QCD colour factors [116]; final states also provide valuable information on the yet poorly understood transition from parton to hadron level (see [117] for a recent review). In the region where an observable's value v (e.g. $1-T$, where T is the transverse thrust in hadron colliders) is small, one should resum logarithmically enhanced contributions that arise at all orders in the perturbative series. For a number of observables such a resummation has been carried out manually at next-to-leading logarithmic (NLL) accuracy [118]. But achieving NLL accuracy requires a detailed analysis of the observable's properties, and is often technically involved. Recently a new approach was proposed [113] based on a general NLL resummed master formula valid for a large class of final-state observables

Under not-overly restrictive conditions, the NLL resummation for the observable's distribution (the probability $\Sigma(v)$ that the observable's value is less than v) for a fixed Born configuration is given by the 'master' formula [113]:

$$\Sigma(v) = e^{-R(v)}\mathscr{F}(R'(v)), \qquad R'(v) = -v\frac{dR(v)}{dv} . \tag{20}$$

The function $R(v)$ is a Sudakov exponent that contains all leading (double) logarithms and all NLL (single-log) terms that can be taken into account by exponentiating the contribution to $\Sigma(v)$ from a single emission. This function depend on a small number of parameters [113].

The advantage of having introduced a master formula is that the resummation of the observable can be performed entirely automatically. The master formula and applicability conditions are encoded in a computer program (CAESAR, Computer Automated Expert Semi-Analytical Resummer), which given only the observable's definition in the form of a computer routine, returns the observable's distribution $\Sigma(v)$ at NLL accuracy (where possible). More details can be found in Refs. [96, 113].

Various other recent developments should be mentioned. In Ref. [119] the formalism for threshold resummation was extended, at NLL accuracy, beyond $2 \to 2$ kinematics to an arbitrary number of partons involved in the hard scattering.

An important new classes of logarithms, so-called non-global logarithms, entering at the NLL level, were identified [120] in observables that are defined to be sensitive to only radiation in only parts of phase space.

Finally, a Lagrangian for near-elastic dynamics, Soft-Collinear Effective Theory, was formulated [121], and has been used to good effect, mostly in the context of B-decays.

CONCLUSIONS

Even though this review of finite order and resummed predictions for hadron collider observables has been brief, and even though we have merely touched upon many developments enabling more accurate predictions, we hope the reader is left with the impression that much has already been achieved, and that much more progress is in the offing. We can look forward to an interesting and, with luck, fruitful confrontation of precision measurements with precision predictions at hadron colliders.

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